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Collineations whose Characteristic Determinants have Linear Elementary Divisors[*] *with an Application to Quadratic Forms.*

By A. B. Coble.[†]

Weirstrass' solution of the problem of the equivalence of two non-singular pencils of bilinear forms is based on the simultaneous reduction of two members of a pencil to a normal or canonical form. Segre[‡] has employed Weirstrass' normal form to obtain a classification of collineations, and has discussed the projective properties of the various classes.

In general, the reduction to this normal form is not unique. A later method, devised by Darboux[||] and perfected by Stickelberger,[§] clearly indicates, by the introduction of systems of parameters, a certain freedom in the reduction. However, the conditions to which the parameters are subjected are of a rather vague nature.

In the following only the general class of collineations mentioned above will be treated. A method will be given for writing down in terms of parameters the *most general* normal form of a collineation of this class. Moreover, the restrictions upon the parameters will appear in very simple form.

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[*] A quite complete account of the literature on this subject is given by Muth—Elementartheiler (Teubner, 1899). Only important references will be given hereafter.

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[‡] Reale Acad. dei Lincei (1884), Ser. 3a, Vol. XIX, p. 6.

[||] Liouville's Journal (1874), Ser. II, Vol. XIX, p. 347.

[§] Crelle's Journal (1879), Vol. 86, p. 20.

SECTION I.

§1.—*Generalities on Collineations.*[*]

The bilinear form

$$A \equiv \sum_{\iota,\kappa} a_{\iota\kappa} x_\iota u_\kappa \equiv (ax)(u\alpha) \equiv (bx)(u\beta) \equiv \ldots . \qquad (\iota, \kappa = 1, 2, \ldots . n)$$

in contragredient variables x and u, which are considered as point and plane[†] coordinates in a linear space of $n-1$ dimensions, S^{n-1}, represents, when equated to zero, two collineations in S^{n-1}, which are inverse. The one collineation is given in the form of a point correspondence, the other in the form of a plane correspondence. As we shall be interested only in points and planes which are unaltered by the collineations,[‡] the distinction between the two is not material.

The number and relative position of the fixed points and planes of A is determined by the behavior of the characteristic determinant of A. We assume first that none of the roots of the characteristic equation are zero, i. e. the collineation is non-singular. If any root substituted in the characteristic determinant makes all minors of degree $n-p$ vanish but not all of degree $n-p-1$, there is a linear point spread of dimensions p, P^p, every point of which is a fixed point. Corresponding to this same root, there is a linear plane spread of dimensions p, Π^p, every plane of which is a fixed plane. P^p and Π^p will be called the fixed spreads of the selected root. No point of the fixed spread of one root lies in the fixed point spread of any other root. A point of the fixed spread of one root lies on *every* plane of the fixed plane spreads of *every other* root. A point of the fixed point-spread of a root may or may not lie on all the planes of the fixed plane-spread of that root.

§2.—*Statement of the Problem.*

Collineations of the kind under consideration may be characterized in various ways:

(1). Their characteristic determinants possess only linear elementary divisors.

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[*] See Segre, loc. cit.

[†] By *plane* is meant a linear spread of dimensions $n-2$ in S^{n-1}.

[‡] Such points and planes will be called "fixed."

(2). No point of the fixed point-spread of any root lies on all the planes of the fixed plane-spread of that same root.

(3). If the characteristic determinant has a root of multiplicity $p+1$, that root makes all the minors of the determinant of degree $n-p$ vanish.

Any one of these three properties necessitates the others. It may easily be shown that the fixed point-spaces of the various roots are not only entirely separate but also *linearly independent.*

The given collineation will be said to be in a normal form when it is expressed by means of n fixed points and n fixed planes, which together form a "basis" in S^{n-1}. In the following the most general normal form will be sought, i. e. one which contains parameters, by particularizing which all possible normal forms are obtained. It is further desirable that the geometric meaning of these parameters and of the conditions to which they are subjected, be evident from the reduction; also, that the reduced form itself shall contain explicitly only the ordinary invariant forms of the collineation (including the roots of the characteristic equation).

§3.—*Combined Parametral Representation of the Spaces P^p and Π^p of a Root of Multiplicity $p+1$.*

Let v be a fixed plane of A. Then v satisfies the identity in x,

$$(ax)(va) \equiv \lambda\,(vx),$$

where λ is a root of the characteristic equation of A. Suppose it is a root of multiplicity $p+1$. By hypothesis, it makes all the minors of degree $n-p$ of the characteristic determinant vanish. $p+1$ equations of the identity may be discarded and $n-p-1$ of the coordinates v_ι solved for in terms of the others. For symmetry, however, a method employed by Hilbert[*] for binary forms will be used. The n equations of the identity are extended by the addition of $p+1$ new variables $\rho, \rho_1, \rho_2, \ldots\ldots \rho_p$ with arbitrary coefficients. The identity

$$(v_1x_1 + v_2x_2 + \ldots\ldots + v_nx_n) - (vx) \equiv 0$$

is added to the system as well as the p relations

$$(vx^1) = 0, \quad (vx^2) = 0, \ldots\ldots (vx^p) = 0.$$

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[*] Math. Annalen (1887), Vol. 28.

The system then reads

$$\begin{array}{lllll}
v_1(a_1\alpha_1-\lambda) & +v_2a_1\alpha_2 & +\ldots+v_na_1\alpha_n & +\rho u_1+\rho_1u_1^1+\ldots+\rho_pu_1^p & =0,\\
v_1a_2\alpha_1 & +v_2(a_2\alpha_2-\lambda) & +\ldots+v_na_2\alpha_n & +\rho u_2+\rho_1u_2^1+\ldots+\rho_pu_2^p & =0,\\
\cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\\
v_1a_n\alpha_1 & +v_2a_n\alpha_2 & \multicolumn{2}{l}{+\ldots+v_n(a_n\alpha_n-\lambda)+\rho u_n+\rho_1u_n^1+\ldots+\rho_pu_n^p} & =0,\\
v_1x_1 & +v_2x_2 & +\ldots+v_nx_n & & =0,\\
 & & & -(vx) & \\
v_1x_1^1 & +v_2x_2^1 & +\ldots+v_nx_n^1 & & =0,\\
\cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\\
v_1x_1^p & +v_2x_2^p & +\ldots+v_nx_n^p & & =0,
\end{array}$$

a system of $n+p+1$ equations in the $n+p+2$ unknowns v_ι, ρ, ρ_ι, (vx). The values of ρ and ρ_ι, as determined from this new system, are zero, since they may be expressed in terms of minors of degree $n-p$. The system as modified above is then equivalent to the original system and the p added relations. Solving for the unknown (vx), we have (the symbol, $\simeq$, being understood to mean "is proportional to"),

$$(vx)\simeq W_p(\lambda)\equiv(-1)^n\begin{vmatrix}
a_1\alpha_1-\lambda & a_1\alpha_2 & \ldots & a_1\alpha_n & u_1 & u_1^1 & \ldots & u_1^p\\
a_2\alpha_1 & a_2\alpha_2-\lambda & \ldots & a_2\alpha_n & u_2 & u_2^1 & \ldots & u_2^p\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
a_n\alpha_1 & a_n\alpha_2 & \ldots & a_n\alpha_n-\lambda & u_n & u_n^1 & \ldots & u_n^p\\
x_1 & x_2 & \ldots & x_n & 0 & 0 & \ldots & 0\\
x_1^1 & x_2^1 & \ldots & x_n^1 & 0 & 0 & \ldots & 0\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
x_1^p & x_2^p & \ldots & x_n^p & 0 & 0 & \ldots & 0
\end{vmatrix}$$

Hence $W_p(\lambda)$ is for arbitrary values $x^1, x^2, \ldots x^p$, the equation of the fixed plane of Π^p which passes through the selected points $x^1, \ldots, x^p$ and this regardless of the values of $u, u^1, \ldots, u^p$ if only they are not such as to make $W_p(\lambda)$ vanish identically. But from the self-dual nature of the result, this theorem follows:

(1). *If* λ *is a* $(p+1)$*-tuple root of the characteristic equation,* $W_p(\lambda)$ *is the product of a fixed point of* P^p *and a fixed plane of* Π^p*, the fixed point lying on the selected planes* $u^1, \ldots, u^p$*; the fixed plane passing through the selected points* $x^1, \ldots, x^p$.

The fixed plane is determined rather by the linear space, S^{p-1}, which passes through the selected points $x^1, \ldots, x^p$. Any linearly independent set of p points of this S^{p-1} will determine the fixed plane as well as the selected set.

$W_p(\lambda)$ contains one set of variables x, u and p sets of parameters $x^1, u^1, \ldots, x^p, u^p$. If the parameters do not appear, the determinant will be denoted by $W(\lambda)$. If also the variables do not appear, the (characteristic) determinant will be denoted by $W_0(\lambda)$. Expanding $W_0(\lambda)$ in powers of λ, we may write

$$W_0(\lambda) \equiv \lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \ldots + c_{n-1}\lambda + c_n.$$

We wish now to obtain the expansions of $W(\lambda)$, $W_1(\lambda), \ldots, W_p(\lambda)$ in terms of c_ι, powers of λ and powers of A. For brevity, the symbolic notation for forms will be used. We put

$$E \equiv A^0 \equiv (ux), \quad A^1 \equiv (ax)(u\alpha), \quad A^2 \equiv (ax)(b\alpha)(u\beta), \quad \text{etc.}$$

Consider the series

$$S \equiv \frac{A^0}{\lambda} + \frac{A^1}{\lambda^2} + \frac{A^2}{\lambda^3} + \ldots$$

which converges for sufficiently large values of λ. Then

$$SA = \frac{A^1}{\lambda} + \frac{A^2}{\lambda^2} + \frac{A^3}{\lambda^3} + \ldots,$$

$$S\lambda E = A^0 + \frac{A^1}{\lambda} + \frac{A^2}{\lambda^2} + \frac{A^3}{\lambda^3} + \ldots,$$

$$\therefore \quad S(\lambda E - A) = A^0 = E$$

and

$$S = (\lambda E - A)^{-1} = \frac{A^0}{\lambda} + \frac{A^1}{\lambda^2} + \ldots = \frac{W(\lambda)}{W_0(\lambda)},$$

$$\therefore \quad W(\lambda) \equiv (\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \ldots + c_n)\left(\frac{A^0}{\lambda} + \frac{A^1}{\lambda^2} + \frac{A^2}{\lambda^3} + \ldots\right).$$

Since the right-hand member is equivalent to a polynomial, it must itself reduce to a polynomial. Therefore, expanding and neglecting negative powers of λ, we have

$$\begin{aligned} W(\lambda) \equiv\ & (\lambda^{n-1} + c_1\lambda^{n-2} + \ldots + c_{n-1})\,A^0 \\ & + (\lambda^{n-2} + c_1\lambda^{n-3} + \ldots + c_{n-2})\,A^1 + \ldots + (\lambda + c_1)\,A^{n-2} + A^{n-1} \end{aligned}$$

If $d_{n-\kappa} \equiv \lambda^{n-\kappa} + c_1\lambda^{n-\kappa-1} + \ldots . + c_{n-\kappa}$,

$$W(\lambda) \equiv d_{n-1}A^0 + d_{n-2}A^1 + d_{n-3}A^2 + \ldots . + A^{n-1}.$$

The same result might be obtained by taking the evectant of $W_0(\lambda)$ as to the coefficients of A, and changing the sign. Comparing this result with the one already obtained, we see that the evectant (with negative sign) of c_1 is A^0; of c_2 is $c_1A^0 + A^1$; of c_3 is $c_2A^0 + c_1A^1 + A^2$, etc.

$W_1(\lambda)$ may now be obtained from $W(\lambda)$ by again taking the negative evectant, new variables x^1 and u^1 being introduced. First, it is evident from the form of $W_1(\lambda)$ that all terms in its expansion will fall into one of two sets, the one set being obtained from the other by permuting x and x^1 and changing sign. Also, the terms must be either of the form $\mu . A\,(x, u) . A^\kappa(x^1, u^1)$ or of the form $\nu A^l(x, u^1) . A^m(x^1, u)$. The sum of terms of the first form and the sum of terms of the second form make up the two sets above mentioned. In taking the evectant of $W(\lambda)$, terms of the first form arise when A^ι is considered constant and c_ι variable; terms of the second form when c_i is considered constant and A^ι variable. Since the result of evection upon c_ι is known, we may write at once

$$W_1(\lambda) \equiv d_{n-2}A^0A^0_{(1)} + d_{n-3}(A^0A^1_{(1)} + A^1A^0_{(1)}) + \ldots . \\ + (A^0A^{n-2}_{(1)} + \ldots . + A^{n-2}A^0_{(1)}) - R_1,$$

where $A^\iota_{(1)}$ is A^ι written with variables x^1 and u^1 and R_1 is the result of permuting x and x^1 in the previous terms.

By a similar argument, we may write

$$W_2(\lambda) \equiv d_{n-3}A^0A^0_{(1)}A^0_{(2)} + d_{n-4}(A^0A^0_{(1)}A^1_{(2)} + A^0A^1_{(1)}A^0_{(2)} + A^1A^0_{(1)}A^0_{(2)}) + \ldots . \pm R_2,$$

where $\pm R_2$ is the result of permuting x, x^1 and x^2 in the previous terms, the sign of the permutation being attached. In general,

$$W_p(\lambda) \equiv d_{n-p-1}\sum_0 + d_{n-p-2}\sum_1 + \ldots . + \sum_{n-p-1} \pm R_p,$$

where $\sum\limits_\kappa$ is the sum of the homogeneous terms of degree κ, composed of powers of A, $A_{(1)}$, $A_{(2)}$, $\ldots .$, $A_{(p)}$ and R_p, has a meaning entirely analogous to that of R_1 and R_2.

§4.—*The Identical Collineation of the Spaces P^p and Π^p.*

Connected with the series of forms $W_1(\lambda), \ldots., W_p(\lambda)$ is a series of covariant collineations obtained by taking the bilinear invariant as to each pair of parameters x^κ and u^κ, the process being repeated until the parameters no longer appear. Manifestly, the same result will be obtained by taking the successive derivatives of $W(\lambda)$ with respect to λ. We define then

$$W^{(\kappa)}(\lambda) \equiv \frac{1}{\varkappa!} \frac{\partial^\kappa}{\partial\lambda^\kappa} W(\lambda).$$

Writing as before

$$W(\lambda) \equiv d_{n-1}A^0 + d_{n-2}A^1 + \ldots. + A^{n-1},$$

we may put

$$\begin{aligned} W^{(1)}(\lambda) &\equiv d^{(1)}_{n-2}A^0 + d^{(1)}_{n-3}A^1 + \ldots. + A^{n-2}, \\ W^{(2)}(\lambda) &\equiv d^{(2)}_{n-3}A^0 + d^{(2)}_{n-4}A^1 + \ldots. + A^{n-3}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ W^{(p)}(\lambda) &\equiv d^{(p)}_{n-p-1}A^0 + d^{(p)}_{n-p-2}A^1 + \ldots. + A^{n-p-1}. \end{aligned}$$

λ being a $(p+1)$-tuple root of $W_0(\lambda)=0$, the various sets of coefficients $d^{(k)}$ have perfectly definite values. The d_ι are the coefficients of an equation which has all the roots of $W_0(\lambda)=0$ except *one* root λ. The $d^{(1)}_\iota$ are the coefficients of an equation which has all the roots of $W_0(\lambda)=0$ except two roots λ, etc. Finally the $d^{(p)}_\iota$ are the coefficients of an equation which has all the roots of $W_0(\lambda)$ except $p+1$ roots λ. Hence, among the sets of coefficients, we have the relations

$$d^{(\rho)}_\kappa = d^{(\rho+1)}_\kappa - \lambda d^{(\rho+1)}_{\kappa-1},$$
$$(\rho = 0, 1, \ldots., p-1)(\varkappa = 1, 2, \ldots., n-\rho).$$

Of these collineations $W(\lambda)$, $W^{(1)}(\lambda), \ldots., W^{(p-1)}(\lambda)$ vanish identically, for they are expressible in terms of minors of degree $n-p$ or greater. With regard to $W^{(p)}(\lambda)$, we have the theorem :

(2). $W^{(p)}(\lambda)$ *is a singular collineation. Its singular point-spread is the linear spread determined by the fixed point-spreads of all the roots other than* λ.

For, let x be a fixed point of A corresponding to the root λ_1. Then the homolog x by $W^{(p)}(\lambda)$ is

$$(d^{(p)}_{n-p-1} + \lambda_1 d^{(p)}_{n-p-2} + \lambda_1^2 d^{(p)}_{n-p-3} + \dots + \lambda_1^{n-p-1})(ux) = 0.$$

Since λ_1 is a root of the equation with coefficients $d^{(p)}_\iota$, the first factor vanishes, i. e. x is a singular point of $W^{(p)}(\lambda)$. Also, it is evident that—

(3). *For the fixed point-spread of the root* λ, $W^{(p)}(\lambda)$ *is the identical collineation.*

Since every point is linearly expressible in terms of all the fixed point-spreads of A, it follows that—

(4). *If the linear spread of dimensions* $n-p-1$ *determined by the fixed point-spreads of all the roots other than* λ, *and a point* x, *cuts* P^p *in a point* x', *then* x' *is the homolog of* x *by* $W^{(p)}(\lambda)$.

$W^{(p)}(\lambda)$ projects, therefore, a point, in general position, upon the fixed spread P^p. Analytically, $W^{(p)}(\lambda)$ may be employed as a transformation which replaces a general point x by a fixed point of P^p. Properties of $W^{(p)}(\lambda)$ dualistic to those stated, may at once be supplied.

§5.—*Transformation of the Determinant* $W_p(\lambda)$.

The parameters $x^1, \dots, x^p$, appearing in $W_p(\lambda)$, have thus far been unrestricted. Instead of selecting these p points arbitrarily, we will now require that they be p points of the spread P^p. $W_p(\lambda)$ will still represent *any* plane of Π^p. For a plane of Π^p cuts P^p in an S^{p-1}, and the p arbitrary points may be selected in this S^{p-1}. This new choice of parameters may e b analytically effected by replacing the variables x^κ by new variables x^κ by means of $W^{(p)}(\lambda)$ according to theorem (4). The effect of this transformation will be followed out in the expanded form of $W_p(\lambda)$.

We suppose, first, that $W_0(\lambda) = 0$ has, besides the root λ of multiplicity $p+1$, the further roots $\lambda_1, \lambda_2, \dots, \lambda_\sigma$ of multiplicity $p_1+1, p_2+1, \dots, p_{\sigma+1}$ respectively. Then

$$\begin{aligned}
W(\lambda) &\equiv (A-\lambda A^0)^p (A-\lambda_1 A^0)^{p_1+1} \dots (A-\lambda_\sigma A^0)^{p_\sigma+1} \equiv (A-\lambda A_0)^p f(A),\\
W^{(1)}(\lambda) &\equiv (A-\lambda A^0)^{p-1} f(A),\\
&\dots\dots\dots\dots\dots\dots\\
W^{(p)}(\lambda) &\equiv f(A).
\end{aligned}$$

We employ now the following theorem due to Frobenius:*

(5). *If* $\psi(\mu)$ *is the* n^{th} *elementary divisor of the characteristic determinant of a bilinear form of* $2n$ *variables, then* $\psi(A) = 0$ *is the equation of lowest degree which the form* A *satisfies.*

Since, in the present case, the elementary divisors are all linear, we have, as the identity of lowest degree which the form A satisfies:

$$(A - \lambda A^0)(A - \lambda_1 A^0)(A - \lambda_2 A^0) \;\ldots\ldots\; (A - \lambda_\sigma A^0) \equiv 0\,.$$

Recalling the value of $f(A)$, we have, as a consequence of this identity:

$$W^{(p)}(\lambda)\,.\,(A - \lambda A^0) \equiv 0,$$

or

$$W^{(p)}(\lambda)\cdot A \equiv \lambda W^{(p)}(\lambda),$$

hence

$$W^{(p)}(\lambda)\,.\,A^2 \equiv \lambda^2 W^{(p)}(\lambda),\ \text{etc.}$$

Arranging the expanded form of $W_p(\lambda)$ in powers of $A_{(p)}$, it takes the form

$$\begin{aligned} A^0_{(p)} \Big[d_{n-p-1} \sum_0 + d_{n-p-2} \sum_1 + \;\ldots\ldots\; + \sum_{n-p-1} \Big] & \\ + A^1_{(p)} \Big[d_{n-p-2} \sum_0 + d_{n-p-3} \sum_1 + \;\ldots\ldots\; + \sum_{n-p-2} \Big] + \;\ldots\ldots & \\ + A^{n-p-2}_{(p)} \Big[d_1 \sum_0 + \sum_1 \Big] + A^{n-p-1}_{(p)} \sum_0 \pm R_p. & \end{aligned}$$

We note at once that in operating on this expression, R_p may be neglected, the result of operation on its term being inferred from the result on the previous terms. Also, where no ambiguity may occur, we write $W^{(p)}(\lambda)$ as $W^{(p)}$.

We replace now the variables x^p in $A^{\iota}_{(p)}$ by operating with variables u in $W^{(p)}(u, x^p)$. According to the above identities, the result is simply $\lambda^{\iota} W^{(p)}(x^p, u^p)$. $W^{(p)}(x^p, u^p)$ will then factor out from the above expression. The remaining factor is

$$\begin{aligned} \Big[d_{n-p-1} \sum_0 + \;\ldots\ldots\; + \sum_{n-p-1} \Big] + \lambda \Big[d_{n-p-2} \sum_0 + \;\ldots\ldots\; + \sum_{n-p-2} \Big] + \;\ldots\ldots & \\ + \lambda^{n-p-2} \Big[d_1 \sum_0 + \sum_1 \Big] + \lambda^{n-p-1} \sum_0 & \end{aligned}$$

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* Crelle's Journal (1878), Vol. 84, p. 12.

in which the coefficient of $\sum\limits_{\iota-1}$ is

$$d_{n-p-\iota} + \lambda d_{n-p-\iota-1} + \dots\dots \lambda^{n-p-} = d^{(1)}_{n-p-\iota},$$

since $d_\kappa = d^{(1)}_\kappa - \lambda d^{(1)}_{\kappa-1}$.

The factor in question is then

$$d^{(1)}_{n-p-1}\sum_0 + d^{(1)}_{n-p-2}\sum_1 + \dots\dots + d^{(1)}_1\sum_{n-p-2} + \sum_{n-p-1}.$$

Proceeding in the same way with this factor, we find, on transforming the parameters x^{p-1}, that $W^{(p)}(x^{p-1}, u^{p-1})$ separates, leaving a new factor

$$d^{(2)}_{n-p-1}\sum_0 + \dots\dots + \sum_{n-p-1}.$$

Finally, we have, as a last factor after transformation of all the parameters

$$d^{(p)}_{n-p-1}A^0 + d^{(p)}_{n-p-2}A^1 + \dots\dots + A^{n-p-1} \equiv W^{(p)}(x, u).$$

The result, then, of the transformation of the parameters in the principal part of $W_p(\lambda)$ is

$$W^{(p)}(x, u) \,.\, W^{(p)}(x^1, u^1) \,.\, W^{(p)}(x^2, u^2) \dots\dots W^{(p)}(x^p, u^p).$$

To obtain the complete result, we permute $x, x^1, \dots\dots, x^p$ in all possible way, prefix the sign of the permutation and add. This result may be expressed thus:

(6). *If in $W_p(\lambda)$ the parameters $x^1, \dots\dots, x^p$ are transformed by means of $W^{(p)}(\lambda)$, the result may be put in the form*

$$V_p(\lambda) \equiv \begin{vmatrix} W^{(p)}(x, u) & W^{(p)}(x^1, u) & \dots\dots & W^{(p)}(x^p, u), \\ W^{(p)}(x, u^1) & W^{(p)}(x^1, u^1) & \dots\dots & W^{(p)}(x^p, u^1), \\ \dots & \dots & \dots & \dots \\ W^{(p)}(x, u^p) & W^{(p)}(x^1, u^p) & \dots\dots & W^{(p)}(x^p, u^p), \end{vmatrix}.$$

§6.—*The Normal Form of $W^{(p)}(\lambda)$.*

In the identity of theorem 6 we assume

$$W^{(p)}(x^\iota, u^\kappa) = 0, \ \iota \neq \kappa \ \text{and} \ W^{(p)}(x^\iota, u^\iota) \neq 0.$$

Let $\prod\limits_1^p{}_\iota W^{(p)}(x^\iota, u^\iota)$ denote the product of the p factors $W^{(p)}(x^1, u^1)$, $W^{(p)}(x^2, u^2)$ .... $W^{(p)}(x^p, u^p)$ and let $\prod\limits_1^p{}_{\iota\kappa} W^{(p)}(x^\iota, u^\iota)$ denote the result of replacing in $\prod\limits_1^p{}_\iota W^{(p)}(x^\iota, u^\iota)$ the factor $W^{(p)}(x^\kappa, u^\kappa)$ by $W^{(p)}(x, u^\kappa)\,.\,W^{(p)}(x^\kappa, u)$, then we may expand the determinant for $V_p(\lambda)$ in the form

$$V_p(\lambda) = W^{(p)}(x, u)\prod_1^p{}_\iota\, W^{(p)}(x^\iota, u^\iota) - \sum_{\kappa=1}^p \prod^p{}_{\iota\kappa}\, W^{(p)}(x^\iota, u^\iota).$$

Solving this equation for $W^{(p)}(x, u) \equiv W^{(p)}(\lambda)$, we have

$$\text{I.} \qquad W^{(p)}(\lambda) \equiv \frac{V_p(\lambda) + \sum\limits_{\kappa=1}^p \prod{}_{\iota\kappa}\, W^{(p)}(x^\iota, u^\iota)}{\prod\limits_1^p{}_\iota\, W^{(p)}(x^\iota, u^\iota)}.$$

$W^{(p)}(\lambda)$ is then expressed in terms of the p points $W^{(p)}(x^\iota, u)$ and the p planes $W^{(p)}(x, u^\iota)$ $(\iota = 1, 2, \ldots., p)$ and the point and plane $V_p(\lambda)$. Each of the p points lies on all the p planes except that which has the same index ι. For the bilinear invariant of $W^{(p)}(x^\iota, u)$ and $W^{(p)}(x, u^\kappa)$ is $[W^{(p)}(x^\iota, u^\kappa)]^2$ or $f(\lambda)\,.\,W^{(p)}(x^\iota, u^\kappa)$. Since $f(\lambda) \neq 0$, this does or does not vanish according as ι is not or is equal to κ. Also, from the above property of the square of $W^{(p)}(\lambda)$, it follows that the result of operating with $W^{(p)}(x^\iota, u)$ or $W^{(p)}(x, u^\iota)$ on $V_p(\lambda)$ vanishes.

We still have to show that under the conditions put upon the parameters, the point and plane given by $V_p(\lambda)$ are not incident, i. e. that the bilinear invariant of $V_p(\lambda)$ is not zero. The bilinear invariant of the p negative terms of V_p is $-p\,.f(\lambda)\,.\prod\limits_1^p{}_\iota W^{(p)}(x^\iota, u^\iota)$; of the positive term is $\prod\limits_1^p{}_\iota W^{(p)}(x^\iota, u^\iota)$ times the bilinear invariant of $W^{(p)}$. Since the bilinear invariant of A^κ is $\Sigma\lambda_\rho^\kappa$, the summation, including all roots, we may write the bilinear invariant of $W^{(p)}(\lambda)$ in the form

$$(p+1)f(\lambda)+(p_1+1)f(\lambda_1)+\ldots\ldots+(p_\sigma+1)f(\lambda_\sigma),$$

of which all terms vanish but the first. Hence the bilinear invariant of $V_p(\lambda)$ is

$$\{(p+1)-p\}f(\lambda)\cdot\prod_{1}^{p}{}_{\iota}\,W^{(p)}(x^\iota,u^\iota)\neq 0,$$

From this follows the theorem :

(7). *The formula* I *gives the most general normal form of* $W^{(p)}(\lambda)$, *the identical collineation in the spaces* P^p *and* Π^p. *The points and planes occuring in the formula are situated in* P^p *and* Π^p *respectively. Points and planes with different indices are incident (the point and plane,* V_p, *being supposed to have the same index). The parameters are subjected only to the conditions*

$$W^{(p)}(x^\iota,u^\varkappa)=0,\ \iota\neq\varkappa\quad and\quad W^{(p)}(x^\iota,u^\iota)\neq 0,\qquad(\iota,\varkappa=1,2,\ldots\ldots,p).$$

§7.—*The Normal Form of the Collineation A.*

Denoting by $f_\iota(\mu)$ a polynomial which has all the roots of $W_0(\mu)$ except the $p_\iota+1$ roots λ_ι, then $f_\iota(A)\equiv W^{(p_\iota)}(\lambda_\iota)$. A treatment entirely similar to that of $W^{(p)}(\lambda)$ will give for each $W_\iota^{(p)}(\lambda_\iota)$ the corresponding normal form. Also put

$$\phi(A)\equiv(A-\lambda A^0)(A-\lambda_1A^0)\ldots\ldots(A-\lambda_\sigma A^0)\equiv 0,$$

$f_\iota(A)$ is a polynomial in A. The highest degree to which A may occur is $n-1$ (in case λ_ι is a simple root). Suppose the highest exponent of A in $f(A)$ or $f_\iota(A)$ is $\varkappa$. To the $\sigma+1$ identities

$$f_\iota(A)\equiv W^{(p_\iota)}(\lambda_\iota),$$

which may be viewed as equations in the unknown $A^0, A^1, A^2, \ldots\ldots, A^\varkappa$, we add that the $\varkappa-\sigma$ identities

$$A^\iota\phi(A)\equiv 0,\qquad \iota=0,1,\ldots\ldots,\varkappa-\sigma-1.$$

The determinant of this system of $\varkappa+1$ non-homogeneous equations in the $\varkappa+1$ unknowns $A^0, \ldots\ldots, A^\varkappa$ is not zero. For the vanishing of this determinant would require the existence of quantities ρ, ρ_1, …. , ρ_κ, not all zero, and such that

$$\rho f(\mu) + \rho_1 f_1(\mu) + \cdots\cdots + \rho_\sigma f_\sigma(\mu) + \rho_{\sigma+1}\phi(\mu) + \cdots\cdots + \rho_\kappa \mu^{\kappa-\sigma-1}\phi(\mu) \equiv 0$$

for all values of μ. But for $\mu = \lambda$, the identity becomes simply $\rho f(\lambda) = 0$ or $\rho = 0$. Similarly, we show that $\rho_\iota = 0$ $(\iota = 1, 2, \ldots\ldots, \sigma)$. The identity must then take the form

$$\rho_{\sigma+1}\phi(\mu) + \rho_{\sigma+2}\mu\phi(\mu) + \cdots\cdots + \rho_\kappa\mu^{\kappa-\sigma-1}\phi(\mu) \equiv 0,$$

which is valid only if $\phi(\mu) \equiv 0$. Hence the determinant of the above system does not vanish. The system may then be solved for the unknown A^1, and we have

$$\text{II.} \qquad A^1 \equiv b W^{(p)}(\lambda) + b_1 W^{(p_\iota)}(\lambda_1) + \cdots\cdots + b_\sigma W^{(p_\sigma)}(\lambda_\sigma),$$

where the b_ι are functions of the roots of the characteristic equation only.

We have then the theorem:

(A). *The formula* II, *in which* $W^{(p_\iota)}(\lambda_\iota)$ *is replaced by its normal form according to formula* I, *is the required general normal form of the collineation* A.

That the n points and planes of this normal form make up a "basis" in S^{n-1} is at once seen from the fact that they are all fixed points and planes and that every fixed point (plane) of one root is incident with every fixed plane (point) of every other root.

The same set of equations gives as well the values of certain powers of A, and in particular of A^0. Solving for A^0, we have

$$\text{III.} \qquad A^0 = b' W^{(p)}(\lambda) + b_1' W^{(p_1)}(\lambda_1) + \cdots\cdots + b_\sigma' W^{(p_\sigma)}(\lambda_\sigma).$$

From formulæ II and III theorem (B) follows:

(B). *The identical collineation in* S^{n-1} *may be expressed in terms of the points and planes which enter into the general normal form of the collineation* A.

In a way this is trivial, since those points and planes form a "basis" in S^{n-1}.

But by the use of theorem (B) we may obtain the normal form of A as given by the usual methods. We will show first that

$$b = \lambda b' \quad \text{and} \quad b_\iota = \lambda_\iota b'_\iota, \qquad \iota = 1, 2, \ldots\ldots, \sigma.$$

Since $$f(A) \equiv W^{(p)}(\lambda), \text{ then } Af(A) \equiv \lambda W^{(p)}(\lambda).$$

Hence the $\sigma + 1$ identities

$$Af_\iota(A) \equiv \lambda_\iota W^{(p_\iota)}(\lambda_\iota),$$

and the $\varkappa - \sigma$ identities

$$A^{\iota+1}\phi(A) \equiv 0 \qquad (\iota = 0, 1, \ldots\ldots, \varkappa - \sigma - 1)$$

form the same system to obtain A as the original system to obtain A^0, except for the additional factor λ_ι in the right-hand members.

$$\therefore \qquad A^1 \equiv b'\lambda W^{(p)}(\lambda) + b'_1\lambda_1 W^{(p_1)}(\lambda_1) + \ldots\ldots + b'_\sigma\lambda_\sigma W^{(p_\sigma)}(\lambda_\sigma).$$

A comparison of this with formula II verifies the above assertion. There follows then

(C). *If the identical collineation in the form* III *be written again as*

$$A^0 \equiv U_1X_1 + U_2X_2 + \ldots\ldots + U_nX_n,$$

the collineation A in the form II *will take the new form*

$$\begin{aligned} A \equiv \lambda\,(U_1X_1 + \ldots\ldots + U_{p+1}X_{p+1}) & \\ + \lambda_1\,(U_{p+2}X_{p+2} + \ldots\ldots + U_{p+p_1+2}X_{p+p_1+2}) + \ldots\ldots & \\ + \lambda_\sigma\,(U_{n-p_\sigma}X_{n-p_\sigma} + \ldots\ldots + U^nX_n). & \end{aligned}$$

This is the normal form as usually written.

The peculiar relations of the minors of the determinant of the above system are worth notice, though easily read off from the above development.

For later use, we consider further the case of $p = n - 1$, i. e. the case of A^1, the identical collineation in S^{n-1}. $W^{(p)}(\lambda)$ vanishes identically; $W_p(\lambda)$ becomes simply the determinant product

$$|x, x^1, x^2, \ldots\ldots, x^p|\,|u, u^1, u^2, \ldots\ldots, u^p|.$$

The transformation (§5) of $W_p(\lambda)$ into the determinant form $V_p(\lambda)$ here becomes simply the multiplication of the determinant and our fundamental formula becomes

$$|x,\, x^1,\, x^2,\, \ldots\ldots,\, x^p|\,|u,\, u^1,\, u^2,\, \ldots\ldots,\, u^p| \equiv \begin{vmatrix} (ux) & (ux^1) & \ldots\ldots & (ux^p) \\ (u^1x) & (u^1x^1) & \ldots\ldots & (u^1x^p) \\ \ldots & \ldots & \ldots\ldots & \ldots \\ (u^px) & (u^p,\, x^1) & \ldots\ldots & (u^p,\, x^p) \end{vmatrix}.$$

Assuming as before $(u_\iota x_\kappa) = 0$, $\iota \neq \varkappa$ and $(u^\iota x^\iota) \neq 0$, also writing $\prod\limits_1^p{}_\iota (u_\iota x_\iota)$ for $(u^1 x^1)(u^2 x^2) \ldots\ldots (u^p x^p)$, but

$$\prod_1^p{}_{\iota\kappa} (u^\iota x^\iota) \text{ for } (u^1x^1) \ldots\ldots (u^{\kappa-1} x^{\kappa-1})(ux^\kappa)(u^\kappa x)(u^{\kappa+1} x^{\kappa+1}) \ldots\ldots (u^p x^p),$$

we may solve for (ux), obtaining

$$(ux) \equiv \frac{|x,\, x^1,\, \ldots\ldots,\, x^p|\,|u,\, u^1,\, \ldots\ldots,\, u^p| + \sum\limits_{\kappa=1}^p \prod\limits_1^p{}_{\iota\kappa} (u^\iota x^\iota_\iota)}{\prod\limits_1^p{}_\iota (u^\iota x^\iota)}$$

This is the expression of the identical collineation (ux) in S^{n-1} in terms of the most general basis of S^{n-1}.

SECTION II.

The Reduction to Normal Form of Non-singular Pencils of Quadratic Forms whose Determinants have only Linear Elementary Divisors.

§1.—*The Collineation Determined by the Groundforms.*

Let

$$(ax)^2 \equiv \sum_{\iota,\,\kappa} a_{\iota\kappa}\, x_\iota\, x_\kappa$$

and

$$(bx)^2 \equiv \sum_{\iota,\,\kappa} b_{\iota\kappa}\, x_\iota\, x_\kappa,$$

where $a_{\iota\kappa} = a_{\kappa\iota}$ and $b_{\iota\kappa} = b_{\kappa\iota}$, $\iota,\, \varkappa = 1,\, 2,\, \ldots\ldots,\, n,$

be two quadratic forms such that the determinant of $(ax)^2 - \lambda\,(bx)^2$ has only linear elementary divisors. We may suppose also

$$|a_{\iota\kappa}| \neq 0 \quad \text{and} \quad |b_{\iota\kappa}| \neq 0.$$

For, if the pencil of forms is non-singular, two non-singular members may be selected as groundforms. Let

$$(u\beta)^2 \equiv \sum_{\iota,\,\kappa} \beta_{\iota\kappa}\, u_\iota\, u_\kappa \qquad \iota, \kappa = 1, 2, \ldots, n$$

be the reciprocal to $(bx)^2$, so that

$$(u\beta)(b\beta)(bx)^2 \equiv (ux).$$

The two groundforms equated to zero represent in S^{n-1} two quadratic surfaces, $(u\beta)^2 = 0$ being the quadratic $(bx)^2 = 0$ in reciprocal coordinates.

Consider now the collineation

$$A \equiv (ax)(a\beta)(u\beta) = 0,$$

which is set up by taking for the homolog of x the polar point as to $(u\beta)^2 = 0$ or $(bx)^2 = 0$ of the polar plane of x as to $(ax)^2 = 0$.

(1). *The characteristic determinant of the collineation A has also only linear elementary divisors.*

For the determinant of the pencil of quadratic forms is that of the pencil of bilinear forms

$$(ax)(ay) - \lambda\,(bx)(by). \tag{2}$$

Transform the variables y by operating upon them with the variables v of $(u\beta)(v\beta)$ and there is obtained the new pencil of bilinear forms

$$\left.\begin{aligned} &(u\beta)(ax)(a\beta) - \lambda\,(b\beta)(bx)(u\beta) \\ \text{or} \qquad &(ax)(a\beta)(u\beta) - \lambda\,(ux). \end{aligned}\right\} \tag{3}$$

Since the pencil (2) may be transformed into ("is equivalent to") the pencil (3), according to a theorem of the theory of forms, the elementary divisors of the pencil (3) coincide with those of the pencil (2).

We suppose, as before, that the characteristic equation of A has $\sigma + 1$ roots $\lambda, \lambda_1, \ldots, \lambda_\sigma$ of multiplicities $p + 1, p_1 + 1, \ldots, p_\sigma + 1$ respectively. Also, we retain for the collineation A and its allied forms the notation of Sec. I.

From the definition of the collineation A and of its fixed planes and points, this theorem follows:

To every fixed point of P^p is coordinated a fixed plane of Π^p, the point and plane being pole and polar as to both $(ax)^2 = 0$ and $(bx)^2 = 0$ or $(u\beta)^2 = 0$. The fixed point and plane will be called "corresponding."

It is this correspondence which differentiates the collineation of the form A from the more general or rather the more arbitrary collineation treated in Sec. I.

§2.—*A Further Transformation of Parameters.*

We have already obtained an expression of the identical collineation A^0* (in the space S^{n-1} to which A refers) in terms of the fixed points and planes of the various roots. For our present purpose a further specialization of the parameters is necessary. We seek to dispose of the parameters which enter into the expression of A^0 so that point and plane of the same index will be *corresponding.*

First the point y and the plane v whose product is given by $W_p(\lambda)$ are such that y lies on the arbitrary planes $u^1, u^2, \ldots, u^p$ and v passes through the arbitrary points $x^1, x^2, \ldots, x^p$. This set of points being chosen we may take u^1 to be the polar plane of x^1, u^2 the polar plane of x^2, etc., all as to $(ax)^2 = 0$. Since y lies on $u^1, u^2, \ldots u^p$ its corresponding plane passes through $x^1, x^2, \ldots x^p$ and being a fixed plane must be v. Hence:

If in $W_p(\lambda)$, u^ι_κ $(\iota = 1, 2, \ldots p;\ \kappa = 1, 2, \ldots n)$ is replaced by $(ax^\iota)\,a_\kappa$, $W_p(\lambda)$ is the product of corresponding point and plane.

This holds whether or not the parameters refer to fixed points, i.e. this transformation of the parameters U^ι_κ may also be carried out on $V_p(\lambda)$ and the result will be the product of corresponding point and plane. The expression of $V_p(\lambda)$ in determinant form in both variables and parameters and may be written

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* Sec. I, §7, Theorem (B).

$$[V_{(p)}(\lambda)]_{u^\iota_\kappa=(ax^\iota)a_\kappa} = \left| \begin{matrix} W^{(p)}(x, u) \dots\dots W^{(p)}(x^p, u) \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ W^{(p)}(x, u^p) \dots\dots W^{(p)}(x^{(p)}, u^{(p)}) \end{matrix} \right|_{u^\iota_\kappa = (ax^\iota)\, a_\kappa} \tag{6}$$

$$\iota = 1, 2, \dots p, \qquad \kappa = 1, 2, \dots n\,.$$

As before we assume

$$\left.\begin{aligned} &[W^{(p)}(x^l, u^m)]_{u^\iota_\kappa = (ax^\iota)\, a_\kappa} = 0 \text{ for } l \neq m, \\ &[W^{(p)}(x^l, u^l)]_{u^\iota_\kappa = (ax^\iota)\, a_\kappa} \neq 0, \qquad (l, m = 1, 2, \dots p). \end{aligned}\right\} \tag{7}$$

And, as before, we may solve for $W^{(p)}(x, u)$.

There remains to show that the other points and planes of the same index, e. g. $W^{(p)}(x^s, u)$ and $[W^{(p)}(x, u^s)_{u^s_\kappa = (ax^s)\, a_\kappa}$ are also corresponding. Now the polar of $W^{(p)}(x^s, u)$ as to $(ax)^2 = 0$ is $[W^{(p)}(x^s, u)]_{u_\kappa = (ax)\, a_\kappa}$. $W^{(p)}(x^s, u)$ may be expressed in powers of A, every power of A in which x is replaced by x^s and u_κ by $(ax)\, a_\kappa$ is symmetrical in x^s and x. Hence

$$[W^{(p)}(x^s, u)]_{u_\kappa = (ax)\, a_\kappa} \equiv [W^{(p)}(x, u^s)]_{u^s_\kappa = (ax^s)\, a_\kappa}, \tag{8}$$

and all points and planes of the same index are corresponding. We have finally then

(9). *The formula I of* §7, *Sec. I, in which the parameters* u^ι_κ *are replaced by* $(ax^\iota)\, a_\kappa$, *gives the most general normal form of* $W^{(p)}(\lambda)$ *for which point and plane of the same index are corresponding. The restrictions upon the parameters are given by* (7).

We will denote $W^{(p)}(\lambda)$ when the parameters have been disposed as described in the theorem by $[W^{(p)}(\lambda)]'$.

§3.—*The Simultaneous Reduction of* $(ax)^2$ *and* $(bx)^2$ *to Sums of Squares.*

We may proceed as in the last paragraph with the other roots and obtain the most general expression of $W^{(p_\iota)}(\lambda_\iota)$ in the form $[W^{(p_\iota)}(\lambda_\iota)]'$. Using the primed forms, we have from theorem (B), §8, Sec. I:

The identical collineation A^0 in S^{n-1} may be put in the form

$$(ux) \equiv b'[W^{(p)}(\lambda)]' + b_1'[W^{(p_1)}(\lambda_1)]' + \dots\dots + b_\sigma'[W^{(p_\sigma)}(\lambda_\sigma)]', \qquad (10)$$

in which every point and plane of the same index are corresponding.

Formula (10) is an identity. We may replace on both sides then u_κ by $(ax)\,a_\kappa$. The left-hand member becomes $(ax)^2$. The right-hand member, as it stands, is a sum of terms of the form $p_\iota\, l_\iota$, where the fixed plane l_ι is the polar of the fixed point p_ι as to $(ax)^2 = 0$. To replace u_κ by $(ax)\,a_\kappa$ is to change p_ι into l_ι. The right-hand member reduces then to a sum of square Σl_ι^2. Let us call the result of this substitution for u in $[W^{(p_\iota)}(\lambda_\iota)]'$, $[W^{(p_\iota)}(\lambda_\iota)]''$. Then is

$$(ax)^2 \equiv \sum_\iota [b_\iota' W^{(p_\iota)}(\lambda_\iota)]'', \qquad \iota = 0, 1, \dots\dots, \sigma.$$

We may also replace in the identity u_κ by $(bx)\,b_\kappa$. On the left-hand side $(bx)^2$ arises. On the right-hand side we obtain terms of the form $l_\iota\,[p_\iota]_{u_\kappa = (bx)\,b_\kappa}$. Since the polar of p_ι as to $(ax)^2 = 0$ is l_ι and the polar of l_ι as to $(u\beta)^2 = 0$ is λp_ι if a term of the root λ is in question, the polar of λp_ι as to $(u\beta)^2 = 0$ is l_ι. Hence,

$$[p_\iota]_{u_\kappa = (bx)\,b_\kappa} = \frac{1}{\lambda}\,[p_\iota]_{u_\kappa = (ax)\,a_\kappa}. \qquad (11)$$

Replacing u_κ by $(bx)\,b_\kappa$ on the left and by $(ax)\,a_\kappa$ on the right, introducing at the same time the factors $\frac{1}{\lambda_\iota}$ to preserve the identity, we have

$$(bx)^2 = \sum_\iota \frac{1}{\lambda_\iota}\,[b_\iota' W^{(p_\iota)}(\lambda_\iota)]'', \qquad \iota = 0, 1, 2, \dots\dots, \sigma.$$

The preceding results may finally be incorporated in the theorem :

(A). *If the formula* III *of* §7, *Sec. I, be applied to the collineation A and* $W^{(p_\iota)}(\lambda_\iota)$, *in which the variables* u_κ *are replaced by* $(ax)\,a_\kappa$ *and every set of parameters* u_κ^l *are replaced by* $(ax^l)\,a_\kappa$, *be denoted by* $[W^{(p_\iota)}(\lambda_\iota)]''$, *then the formulae* and

$$\left.\begin{aligned}(ax)^2 &\equiv \sum_\iota b'_\iota [W^{(p_\iota)}(\lambda_\iota)]'',\\ (bx)^2 &\equiv \sum_\iota \frac{1}{\lambda_\iota} b'_\iota [W^{(p_\iota)}(\lambda_\iota)]''\end{aligned}\right\} \tag{12}$$

give the most general simultaneous reduction of the two quadratic forms to sums of squares. The parameters are subjected only to the conditions

$$[W^{(p_\iota)}(x^l, u^m)]_{u^m_k = (ax^m)a_\varkappa} \left.\begin{matrix} = 0 \text{ if } l \neq m \\ \neq 0 \text{ if } l = m \end{matrix}\right. \left\{\begin{aligned} \iota &= 0,\ 1,\ 2, \ldots, \sigma,\\ \varkappa &= 1,\ 2, \ldots, n,\\ l,\ m &= 1,\ 2, \ldots, p_\iota.\end{aligned}\right\} \tag{13}$$

§4.—*Related Geometric Facts.*

Two quadrics such that the pencil of quadratic forms determined by them is of the sort treated, possess geometric relations of considerable interest. For example, our present case may be characterized as follows:

(14). *If the two quadrics in S^{n-1}, $(ax)^2 = 0$ and $(bx)^2 = 0$, have contact at any point, they have contact on a quadric (in general of dimensions less than $n - 2$) which contains the point:*

First, if all the roots of the characteristic equation are distinct, there is no contact, for contact at a point requires the incidence of corresponding fixed point and plane, and, therefore, the existence of a double root at least. But for a root of multiplicity $p + 1$, the linear spread P^p cuts either quadric in a quadric of dimension $p - 1$, say Q^{p-1}. But, by definition, if a fixed point lies on one quadric, it lies on both, and they touch at this point. The two groundforms will then touch along Q^{p-1}.

(15). *The quadric $[W^{(p_\iota)}(\lambda_\iota)]_{u_\varkappa = (ax)a_\varkappa} = 0$ has for double points all points of the linear spread determined by the fixed spreads of the roots other than λ_ι. It touches $(ax)^2 = 0$ and $(bx)^2 = 0$ along the quadric of contact $Q^{p_\iota - 1}$.*

The polar system of the quadric

$$[W^{(p_\iota)}(\lambda_\iota)]_{u_\kappa=(ax)\,a_\kappa}=0 \tag{16}$$

is given by

$$[W^{(p_\iota)}(\lambda_\iota)]_{u_\kappa=(ay)\,a_\kappa}=0, \tag{17}$$

for, according to (8), the form (17) is symmetrical in x and y. This bilinear form (17) gives for a point x a plane which is the polar plane as to $(ax)^2=0$ of the homolog of x by $W^{(p_\iota)}(\lambda_\iota)$. Hence, every singular point of $W^{(p_\iota)}(\lambda_\iota)$ is a double point of the quadric (15). Since $(ax)^2=0$ is non-singular, no other singular points exist. Also, if x is a point of $Q^{p_\iota-1}$, its homolog by $W^{(p_\iota)}(\lambda_\iota)$ is again x. The polar as to $(ax)^2=0$ is then the tangent plane of (15) and $(ax)^2=0$.

The quadric (15) contains the whole linear spread determined by a point of $Q^{p_\iota-1}$ and all the fixed point-spreads of the roots other than λ_ι. Since the tangent planes to $(ax)^2=0$ at points of $Q^{p_\iota-1}$ also pass through the same fixed spreads and are the only tangent planes which do so, we may say:

(18). *The quadric* $[W^{(p_\iota)}(\lambda_\iota)]_{u_\kappa=(ax)\,a_\kappa}=0$ *is the envelope of the common tangent planes of* $(ax)^2=0$ *and* $(bx)^2=0$ *at their points of contact on* $Q^{p_\iota-1}$.

The conditions upon the parameters in the reduction of theorem (A), §3, have now a simple geometric meaning. Speaking only of the parameters entering in $[W^{(p_\iota)}(\lambda_\iota)]''$, first x^1 may be chosen at random; x^2 at random upon the polar plane of x^1 as to the quadric (15); x^3 at random upon the polar planes of both x^1 and x^2 as to (15), etc. Finally, x^{p_ι} is chosen at random upon the polar planes of $x^1, x^2, \ldots\ldots, x^{p_\iota-1}$ as to (15).

§5.—*Reduction of a Single Quadratic Form to a Sum of Squares.*

We have obtained in §7, Sec. I, when A itself reduces to the identical collineation, an expression of A in terms of the most general basis in S^{n-1}. Under our present hypotheses, A becomes the identical collineation only when $(ax)^2=0$ and $(bx)^2=0$ coincide. Supposing then $(ax)^2\equiv(bx)^2$, we put the resulting collineation in its most general normal form, and again replace u_κ by $(ax)\,a_\kappa$ and

u^{ι}_{κ} by $(ax^{\iota})\,a_{\kappa}$. We have then the most general expression of $(ax)^2 = 0$ as a sum of the squares of planes. The condition upon the parameters are

$$(ax^{\iota})(ax^{\kappa})\left\{\begin{matrix} = 0 \text{ for } \iota \neq \varkappa, \\ \neq 0 \text{ for } \iota = \varkappa, \end{matrix}\right\} \qquad \iota,\ \varkappa = 1,\ 2,\ \ldots\ldots,\ n-1.$$

This is, of course, only the well-known expression of a quadric in S^{n-1} by means of a "self-polar" basis.

BALTIMORE, *Nov.* 1903.